A THEORY OF ELASTICITY WITH SPATIAL DISTRIBUTION OF MATTER. THREE-DIMENSIONAL COMPLEX STRUCTURE

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References [1 and 2] consider a theory of elasticity with spatial distribution of matter for a medium having simple structure and for a one-dimensional medium having complex structure. In the present article the general case of a three-dimensional medium with complex structure is examined. The general scheme of the one-dimensional case [2] is retained; chief attention is directed toward the specific character of the three-dimensional problem. The original micro-model is a complex crystal lattice [3]. In Section 1 this model is generalized to the case of a continuous distribution of matter. The displacements of the mass centers of the unit cells and the micro-strains of the cells are introduced as the kinematic variables. The force variables are the micro-moments. The transition to an exact continuous representation is carried out, and the equations of an elastic medium of complex structure with spatial distribution of matter are derived. The operators corresponding to the continuous theory are expressed in terms of the original micro-parameters. It is shown that the well known conditions of symmetry of the tensor of elastic constants, which are usually interpreted as the condition of absence of initial stresses [3 and 4], are consequences of the invariance of the elastic energy under translation and rotation. In Section 2 some special models are examined, and the equations of a medium are obtained for the approximation of weak dispersion of matter. These equations contain as a special case the equations of linear nonsymmetric elasticity (couple-stress theory) [5 to 7]. However, in the latter it turns out that the orders of approximation are inconsistent in the various equations from the point of view of the theory of spatial distribution.

In Section 3 the equations of a medium having complex structure are transformed in the acoustic range into equations, one of which contains only a single kinematic variable (the displacement of the mass centers) and the others of which are explicitly solvable for the remaining kinematic variables. The first equation of this set coincides in form with the equation for a medium with simple structure, but differs from it by the presence of a time-wise dispersion which is unrelated to energy dissipation. Expressions are written for the energy density, and it is shown that it is possible to introduce a symmetric stress tensor, as in the case of a simple structure.

1. We shall consider an unbounded complex crystal lattice in the harmonic approximation [3] as the original micro-model of a medium with complex structure. A geometrically complex latice is a three-dimensional periodic structure with a unit cell constructed on the base vectors e_α ($\alpha = 1, 2, 3$).

The positions of the N particles with masses m_j $(j = 1, \ldots, N)$ in the unit cell are specified by the choice of the vectors ξ_j. The interactions between the particles are determined by the force constants $\Phi^{\alpha\beta}(n - n', j, j') = \Phi^{\beta\alpha}(n' - n, j', j)$, where n is a vector characterizing the number of the unit cell. In an oblique coordinate system x^α with base e_α and a metric tensor $g_{\alpha\beta} = e_\alpha \cdot e_\beta$, the components of the vector n are integers.

In the absence of initial forces, the Lagrangian of the complex lattice has the form (the summation convention is used for repeated tensorial indices)

$$2L = g^{\alpha\beta}\sum_{nj} m_j \dot{w}_\alpha(n, j)\dot{w}_\beta(n, j) - \sum_{nn'jj'} w_\alpha(n, j)\Phi^{\alpha\beta}(n - n', j, j')w_\beta(n', j') + \\ + 2\sum_{nj} w_\alpha(n, j) f^\alpha(n, j) \tag{1.1}$$

where $w_\alpha(n, j)$ is the displacement of the jth particle in the nth cell, $f^\alpha(n,j)$ is the external force acting on that particle.

We shall generalize the present model so as to include in our considerations a continuous as well as a discrete periodic structure. To do this we introduce the density $\rho(\xi)$ which characterizes the distribution of mass within a unit cell and the force matrix $\Phi^{\alpha\beta}(n, \xi, \xi') = \Phi^{\beta\alpha}(-n, \xi', \xi)$, which determines the interaction. Here ξ^α is a local system of coordinates with origin at the mass center of the unit cell. The Lagrangian may then be written in the form

$$2L = g^{\alpha\beta}\sum_n \int \rho(\xi)\dot{w}_\alpha(n, \xi)\dot{w}_\beta(n, \xi)\,d\xi - \tag{1.2}$$

$$- \sum_{nn'} \iint w_\alpha(n, \xi)\Phi^{\alpha\beta}(n - n', \xi, \xi')w_\beta(n', \xi')\,d\xi\,d\xi' + 2\sum_n \int w_\alpha(n, \xi) f^\alpha(n, \xi)\,d\xi$$

where, as in [1 and 2], the remote effects are assumed to be bounded, i.e. $\Phi^{\alpha\beta}(n, \xi, \xi')$ is nonzero only in a finite region of values of n. (1.3)

To make the transition to a discrete structure, we set

$$\rho(\xi) = \sum_j m_j \delta(\xi - \xi_j), \qquad \Phi^{\alpha\beta}(n, \xi, \xi') = \sum_{jj'} \Phi^{\alpha\beta}(n, j, j')\,\delta(\xi - \xi_j)\,\delta(\xi' - \xi_{j'})$$

Using the algorithm given in [1] we transform from functions of the discrete argument n in Equation (1.2) to functions of a continuous variable x. In the (k, ω) representation (*) the expression for the Lagrangian assumes the form

$$16\pi^3 v L = \omega^2 g^{\alpha\beta}\iint \rho(\xi)\overline{w_\alpha(k, \xi)}w_\beta(k, \xi)\,dk\,d\xi - \tag{1.4}$$

$$- \iiint \overline{w_\alpha(k, \xi)}\Phi^{\alpha\beta}(k, \xi, \xi')w_\beta(k, \xi')\,dk\,d\xi\,d\xi' + 2\iint \overline{w_\alpha(k, \xi)} f^\alpha(k, \xi)\,dk\,d\xi$$

where v is the volume of the unit cell.

As in the case of one-dimensional structure [2], it is convenient to

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*) Here, as in [1], the functions of k and ω are the Fourier transforms of the corresponding functions of x and t, where $k \in B$, B being the cell of the reciprocal lattice with identification of the points of opposite faces. The dependence on the argument is not indicated explicitly.

introduce collective cell variables in order to make the transition to the theory of elasticity. To this end, we define the moment of inertia tensor of order s of a cell ($s = 0, 1, 2, \ldots$)

$$\rho^{\lambda_1 \ldots \lambda_s} = \frac{1}{v} \int \rho(\xi)\, \xi^{\lambda_1} \ldots \xi^{\lambda_s}\, d\xi \qquad \text{for} \qquad \rho^s = (\rho, \xi^s) \tag{1.5}$$

where s denotes a collective tensorial index in the condensed notation and the parentheses denote a scalar product.

We denote the tensor which is the inverse of ρ^{2s} by ρ_{2s}^{-1} and introduce the two quasidiogonal matrices

$$I^{rs} = \rho^{2s}\delta^{rs}, \qquad I_{rs}^{-1} = \rho_{2s}^{-1}\delta_{rs} \tag{1.6}$$

With the aid of the well known algorithm we construct an orthonormal system of base polynomials $e^s(\xi) \equiv e^{\lambda_1 \ldots \lambda_s}(\xi)$ with weight $\rho(\xi)$ and with the associated system of base functions $e_r(\xi) \equiv e_{\mu_1 \ldots \mu_s}(\xi)$, defined by the relations

$$(\rho e^r,\ e^s) = I^{rs}, \qquad (e_r,\ e^s) = \delta_r{}^s \tag{1.7}$$

It is easily shown that

$$e_r(\xi) = \rho(\xi)\, I_{rs}^{-1} e^s(\xi) \tag{1.8}$$

Considering that the origin of the local system of coordinates coincides with the mass center of the unit cell, we have for the first two elements of the base

$$e^0(\xi) = 1, \quad e^\lambda(\xi) = \xi^\lambda; \qquad e_0(\xi) = \rho_0{}^{-1}\rho(\xi), \quad e_\mu(\xi) = \rho_{\mu\lambda}^{-1}\, \xi^\lambda\, \rho(\xi) \tag{1.9}$$

We now expand the functions of ξ and ξ' which occur in Equation (1.4) in the base elements

$$w_\beta(k, \xi) = w_{s\beta}(k)\, e^s(\xi), \quad f^\alpha(k, \xi) = f^{r\alpha}(k)\, e_r(\xi)$$

$$\Phi^{\alpha\beta}(k, \xi, \xi') = \Phi^{r\alpha s\beta}(k)\, e_r(\xi)\, e_s(\xi') \tag{1.10}$$

Using (1.7), we may easily show that

$$w_{s\beta}(k) = (e_s, w_\beta), \qquad f^{r\alpha}(k) = (e^r, f^\alpha)$$

$$\Phi^{r\alpha s\beta}(k) = \frac{1}{v^2} \iint \Phi^{\alpha\beta}(k, \xi, \xi')\, e^r(\xi)\, e^s(\xi')\, d\xi\, d\xi' \tag{1.11}$$

In the notation of [1] and in the new variables, the Lagrangian (1.4) takes the form ($I^{r\alpha s\beta} = I^{rs} g^{\alpha\beta}$)

$$2L = \langle w_{r\alpha} | \omega^2 I^{r\alpha s\beta} | w_{s\beta} \rangle - \langle w_{r\alpha} | \Phi^{r\alpha s\beta} | w_{s\beta} \rangle + 2 \langle w_{r\alpha} | f^{r\alpha} \rangle \tag{1.12}$$

Taking Equation (1.9) into account, we can give a simple interpretation to the first coefficients of the expansions of the functions w and f. It is easy to see that $w_{0\beta}$ is the displacement of the mass center of the unit cell, and that $f^{0\alpha}$ is the mean density of the body forces. These quantities must obviously have special significance for a macroscopic description of the medium. Because of this, we shall introduce special notations for them, setting $u_\beta = w_{0\beta}$, $q^\alpha = f^{0\alpha}$. The remaining kinematic and force variables will be denoted by $\eta_{q\beta} = w_{q\beta}$ and $\mu^{p\alpha} = f^{p\alpha}$ ($p, q = 1, 2, \ldots$). It follows from (1.9) that the representation $\eta_{\mu\beta} = \varepsilon_{(\mu\beta)'} + \Omega_{[\mu\beta]'}$, is valid,

where $\varepsilon_{(\mu\beta)}$ is the mean strain of the unit cell and $\Omega_{[\mu\beta]}$ is the mean rotation of the cell. Analogously, $\mu^{\lambda\alpha} = \mu^{(\lambda\alpha)} + m^{[\lambda\alpha]}$, where $\mu^{(\lambda\lambda)}$ is the mean density of the force dipoles, and $m^{[\lambda\alpha]}$ is the mean density of moments. The quantities $\eta_{q\beta}$ and $\mu^{p\alpha}$ for $p, q > 1$ represent micro-strains and micro-moments of higher order.

We note that the number of independent base elements is determined by the number of degrees of freedom of the unit cell. If the number of degrees of freedom does not exceed twelve (in the case of discrete structure this means that the number of particles in the cell is not greater than four), the base is automatically limited to the first two elements. In this case the kinematic parameters are the displacement of the mass center u, the micro-strain ε', and the micro-rotation Ω'. Correspondingly, the force parameters are the body-force density q, the micro-moments of the force dipoles μ, and the micro-moments m.

Let us now consider the conditions which are imposed on $\Phi^{r\alpha s\beta}(k)$ by the requirements of invariance of the strain energy

$$\Phi = \tfrac{1}{2} \langle w_{r\alpha} | \Phi^{r\alpha s\beta} | w_{s\beta} \rangle \tag{1.13}$$

under translation and rotation. Let the displacements corresponding to these be $w_\alpha^*(k, \xi)$. Then for any $w_{r\alpha}$ the equation $\Phi(w_{r\alpha} + w_{r\alpha}^*) = \Phi(w_{r\alpha})$ must be satisfied. It follows from this that

$$\operatorname{Re} \langle w_{r\alpha}^* | \Phi^{r\alpha s\beta} | w_{s\beta} \rangle = 0, \qquad \langle w_{r\alpha}^* | \Phi^{r\alpha s\beta} | w_{s\beta}^* \rangle = 0 \tag{1.14}$$

For a translation by the vector a_α, the displacement $w_\alpha^*(k, \xi) \sim a_\alpha \delta(k)$ and, therefore, $w_{r\alpha}^*(k) \sim a_\alpha \delta_r{}^0 \delta(k)$. For the rotation defined by the tensor $a_{\alpha\lambda} = -a_{\lambda\alpha}$, we have $w_\alpha^*(k, \xi) \sim a_{\alpha\lambda}(\xi^\lambda - i\partial^\lambda)\delta(k)$ and, therefore,

$$w_{r\alpha}^*(k) \sim a_{\alpha\lambda}(\delta_r{}^\lambda - i\,\delta_r{}^0 \partial^\lambda)\,\delta(k), \quad \partial^\lambda = \partial^\lambda / \partial k_\lambda$$

From the first condition of (1.14), we find for the case of translation

$$\Phi_0{}^{0\alpha s\beta} = \Phi_0{}^{s\beta 0\alpha} = 0 \tag{1.15}$$

Here and in what follows Φ_0 denotes the value of Φ for $k = 0$. The same condition gives for the case of rotation

$$\Phi_0^{[\lambda\alpha]s\beta} = i\partial^{[\lambda}\Phi_0^{0\alpha]s\beta} \tag{1.16}$$

We now take account of the fact that, by virtue of the conditions $\Phi^{\alpha\beta}(n, \xi, \xi') = \Phi^{\beta\alpha}(-n, \xi', \xi)$ the tensor $\Phi^{r\alpha s\beta}(k)$ is Hermitian, i.e.

$$\Phi^{r\alpha s\beta}(k) = \Phi^{+r\alpha s\beta}(k) \equiv \overline{\Phi^{s\beta r\alpha}(k)} = \Phi^{s\beta r\alpha}(-k) \tag{1.17}$$

where the cross denotes the transposed conjugate with respect to the indices $r\alpha$ and $s\beta$. It then follows from (1.15) and (1.16) that

$$\partial^\lambda \Phi_0^{0\alpha 0\beta} = 0 \tag{1.18}$$

Thus, the following representation for the matrix Φ is valid:

$$\Phi(k) = \left\| \begin{array}{c|c} k_\lambda k_\mu \gamma^{\lambda\alpha\mu\beta}(k) & ik_\lambda \chi^{+\lambda\alpha q\beta}(k) \\ \hline -ik_\mu \chi^{p\alpha\mu\beta}(k) & \Gamma^{p\alpha q\beta}(k) \end{array} \right\| \tag{1.19}$$

where $\gamma(k) = \gamma^+(k)$, $\chi(k)$, $\Gamma(k) = \Gamma^+(k)$ are entire functions which are determined uniquely by the specification of the original matrix $\Phi(k)$. It follows then from (1.16) that

$$\chi_0^{p\alpha[\mu\beta]} + \Gamma_0^{p\alpha[\mu\beta]} = 0 \tag{1.20}$$

Turning to the second condition of (1.14), we note that the direct substitution of $w_{r\alpha}^*$ in this expression is not possible. This is related to the fact that the functional $\Phi(w_{r\alpha})$, given in Equation (1.13) is, in general, undefined for displacements which do not vanish at infinity in the x-space. It is, therefore, expedient in the present case to introduce a strain energy density which must be invariant under translation and rotation. It follows from the form of the function $w_{r\alpha}^*(k)$ that the requirements of invariance impose conditions only on γ_0, χ_0 and Γ_0 and it is, therefore, sufficient to restrict our consideration to the energy density in the zeroth approximation in k, i.e. to the energy density for a homogeneous deformation. Taking Equation (1.19) into account, we can represent the zeroth approximation for the density of the Lagrangian in the form (1.21)

$$2\varphi_0'(x) = \partial_\lambda u_\alpha(x)\gamma_0^{\lambda\alpha\mu\beta}\partial_\mu u_\beta(x) + 2\eta_{p\alpha}(x)\chi_0^{p\alpha\mu\beta}\partial_\mu u_\beta(x) + \eta_{p\alpha}(x)\Gamma_0^{p\alpha q\beta}\eta_{q\beta}(x)$$

which is already invariant under translation. The correct energy density $\varphi_0(x)$ differs from $\varphi_0'(x)$ by divergent terms which must be chosen from the condition of the invariance of $\varphi_0(x)$ under rotation. It is easy to show that the most general divergent part which is invariant under translation has the form

$$\varphi_0(x) - \varphi_0'(x) = \partial_\lambda [u_\alpha(x) b^{\lambda\alpha\mu\beta}\partial_\mu u_\beta(x)] \tag{1.22}$$

where $b^{\lambda\alpha\mu\beta} = -b^{\mu\alpha\lambda\beta}$ is a constant tensor. It is necessary and sufficient for the invariance of $\varphi_0(x)$ under rotation to require that

$$[b^{\lambda\alpha\mu\beta} + b^{\mu\beta\lambda\alpha} + \gamma_0^{\lambda\alpha\mu\beta} + \chi_0^{\mu\beta\lambda\alpha}]_{[\mu\beta]} = 0 \tag{1.23}$$

It can be shown that the condition of solvability of Equation (1.23) is

$$[\alpha\beta, \lambda\mu] = [\lambda\mu, \alpha\beta] \tag{1.24}$$

where

$$[\alpha\beta, \lambda\mu] = \gamma_0^{\lambda\alpha\mu\beta} + [\chi_0^{\lambda\alpha\mu\beta} + \chi_0^{\mu\beta\lambda\alpha} + \Gamma_0^{\lambda\alpha\mu\beta}]_{(\lambda\mu)} \tag{1.25}$$

The solution of (1.23) is unique under the condition (1.24) and has the form

$$2b^{\lambda\alpha\mu\beta} = [2\gamma_0^{\alpha\lambda\mu\beta} - \chi_0^{\lambda\mu\alpha\beta} - \chi_0^{\lambda\alpha\mu\beta} + \chi_0^{\alpha\lambda\mu\beta}]_{[\lambda\mu]} \tag{1.26}$$

In the case of a discrete structure, Equation (1.24) coincides with Huang's condition on the force constants, which in [3 and 4] is interpreted as the condition of absence of initial stresses. From the considerations given in [1] and in the present article, it follows that this condition is not related to initial stresses but is a consequence of the invariance of the energy density under rotation.

The condition of invariance of the energy under rotation can be interpreted heuristically if the macroscopic rotation $\Omega_{\mu\beta} = \partial_{[\mu} u_{\beta]}$ is introduced along with the micro-rotation $\Omega_{\mu\beta}'$. Then it is easy to show that the condition obtained above is equivalent to the requirement that the energy

density in the zeroth approximation depend only on the difference $\Omega - \Omega'$.

Taking account of (1.19), the equations of motion corresponding to the Lagrangian (1.12) have the form

$$\begin{aligned} \omega^2\rho g^{\alpha\beta}u_\beta - k_\lambda\gamma^{\lambda\alpha\mu\beta}(k)\,k_\mu u_\beta - ik_\lambda\chi^{+\lambda\alpha q\beta}(k)\,\eta_{q\beta} &= -q^\alpha \\ \omega^2 I^{p\alpha q\beta}\eta_{q\beta} + i\chi^{p\alpha\mu\beta}(k)\,k_\mu u_\beta - \Gamma^{p\alpha q\beta}(k)\,\eta_{q\beta} &= -\mu^{p\alpha} \end{aligned} \tag{1.27}$$

in the (k, ω) representation, where ρ is the mean mass density in the unit cell. In the (x, t) representation, Equations (1.27) assume the form

$$\begin{aligned} \rho g^{\alpha\beta}\ddot{u}_\beta - \partial_\lambda\gamma^{\lambda\alpha\mu\beta}\partial_\mu u_\beta - \partial_\lambda\chi^{+\lambda\alpha q\beta}\eta_{q\beta} &= q^\alpha \\ I^{p\alpha q\beta}\ddot{\eta}_{q\beta} + \chi^{p\alpha\mu\beta}\partial_\mu u_\beta + \Gamma^{p\alpha q\beta}\eta_{q\beta} &= \mu^{p\alpha} \end{aligned} \tag{1.28}$$

where γ, χ, Γ are the integral operators with kernels of difference form which are the inverse Fourier transforms of the functions $\gamma(k)$, $\chi(k)$ and $\Gamma(k)$.

Equations (1.28) provide an exact continuous representation of the original model. The operators γ, χ, Γ are expressible explicitly in terms of the microscopic force constants. From the phenomenological point of view, these equations describe the most general model of a macroscopically homogeneous, linearly elastic medium of complex structure with a spatial distribution of matter (*).

Just as in the one-dimensional case [2], the model under consideration can be described equivalently in terms of a single kinematic variable $w_\alpha(x)$ and a single force variable $f^\alpha(x)$. Here the equations of motion in both the (k, ω) and (x, t) representations have the form of integral equations with kernels of difference type. The equations which establish the one-to-one correspondence between this description and the one considered above are obtained by the obvious generalization of the corresponding equations for the one-dimensional case, and are, therefore, not given here.

2. Let us consider some special cases. If the unit cell has a center of symmetry, then certain conditions are imposed on the force constants. In matrix notation these conditions coincide with the corresponding conditions obtained for the one-dimensional model [2]. Likewise, it is easy to generalize to the three-dimensional case the necessary and sufficient conditions given in [2] for which the equations of a medium with complex structure admit a transformation to the equations of a medium with simple structure [1].

In the general case, $\Phi^{r\alpha s\beta}(k)$ does not possess symmetry with respect to $\alpha\beta$. However, if this symmetry is present, it can be given a simple interpretation in terms of paired interaction. We term an interaction in a discrete model paired if the potential energy can be represented in the form of a sum, each term of which depends only on the relative displacements of two particles. The generalization of this definition to the case of a continuous model is obvious. It is shown in [10] that in the harmonic approximation the most general form of interaction in a crystal lattice is a triple interaction. It can be proved that the symmetry of $\Phi^{\alpha\beta}(n, \xi, \xi')$ with respect to $\alpha\beta$, and, therefore, of $\Phi^{r\alpha s\beta}(k)$, as well, is a necessary and sufficient condition for the interactions to be paired. It follows from this that in the one-dimensional model the interaction is always paired. In three-dimensional models

*) We remark that a medium of complex structure was considered from another point of view in [7 to 9].

with simple structure, invariance under inversion is usually assumed, and the symmetry of $\Phi^{\alpha\beta}(n)$ with respect to $\alpha\beta$ follows, i.e. the interaction is paired.

We note the interesting case in which the medium has a structure such that for a homogeneous static strain, external forces acting on each particle are absent (this happens, e.g. if each particle is at a center of symmetry). In this case we find a necessary and sufficient condition on the elastic constants from Equations (1.27)

$$\chi_0^{p\alpha(\mu\beta)} + \Gamma_0^{p\alpha(\mu\beta)} = 0 \tag{2.1}$$

Together with (1.20) this means that

$$\chi_0^{p\alpha\mu\beta} + \Gamma_0^{p\alpha\mu\beta} = 0 \tag{2.2}$$

Let us now consider a model of a medium with weak distribution of matter. To do this, we expand the functions $\gamma(k)$, $\chi(k)$ and $\Gamma(k)$ which appear in (1.27) in the series in k in the neighborhood of the point $k = 0$ and restrict ourselves to a finite number of terms. For $k = 0$ we have the zeroth approximation. To go on to further approximations it is useful to distinguish between two cases. For a gyrotropic medium, the next approximation is the first, and the operators in (1.28) have the form (2.3)

$$\gamma^{\lambda\alpha\mu\beta} = \gamma_0^{\lambda\alpha\mu\beta} + \gamma_1^{\lambda\alpha\mu\beta\nu}\partial_\nu, \quad \chi^{p\alpha\mu\beta} = \chi_0^{p\alpha\mu\beta} + \chi_1^{p\alpha\mu\beta\nu}\partial_\nu, \quad \Gamma^{p\alpha q\beta} = \Gamma_0^{p\alpha q\beta} + \Gamma_1^{p\alpha q\beta\nu}\partial_\nu$$

If the medium is not gyrotropic, the next approximation after the zeroth is the second, and

$$\gamma^{\lambda\alpha\mu\beta} = \gamma_0^{\lambda\alpha\mu\beta} + \gamma_2^{\lambda\alpha\mu\beta\nu\tau}\partial_\nu\partial_\tau, \quad \chi^{p\alpha\mu\beta} = \chi_0^{p\alpha\mu\beta} + \chi_2^{p\alpha\mu\beta\nu\tau}\partial_\nu\partial_\tau$$
$$\Gamma^{p\alpha q\beta} = \Gamma_0^{p\alpha q\beta} + \Gamma_2^{p\alpha q\beta\nu\tau}\partial_\nu\partial_\tau \tag{2.4}$$

The subscripts 1 and 2 in (2.3) and (2.4) denote the coefficients in the expansions of the functions $\gamma(k)$, $\chi(k)$ and $\Gamma(k)$ in series of powers of $(-ik)$.

In the particular case of an isotropic medium with the kinematic variables u and Ω' (a Cosserat continuum), Equations (1.28) in the second approximation are written in the form

$$\rho u'' - (\alpha_0 + \alpha_2\Delta)\,\Delta u - (\beta_0 + \beta_2\Delta)\,\text{grad div}\, u + 2(\chi_0 + \chi_2\Delta)\,\text{rot}\,\Omega' = q$$
$$I\Omega'' + (\chi_0 + \chi_2\Delta)\,\text{rot}\, u - 2(\chi_0 - \varkappa_2\Delta)\,\Omega' = m \tag{2.5}$$

where the constants $\alpha_0, \alpha_2, \ldots$ are related in an obvious fashion to the coefficients of the operators of (2.4) in the isotropic case.

The equations of a medium with a weak distribution of matter contain as a special case the equations of the linear nonsymmetric (couple-stress) theory of elasticity [5 to 7]. Thus, for example, in order to obtain the equations of [7] it is necessary to consider a model in which the kinematic variables are u, ε' and Ω'; to limit oneself to the zeroth approximation in all variables in the first equation of the system (1.28); but to take in the second equation the zeroth approximation for u and the second approximation for ε' and Ω'. Such an inconsistency in the approximations cannot, of course, be justified within the framework of a theory of elasticity of a medium with spatial distribution of matter. In [5 and 6] a Cosserat continuum is considered with an analogous inconsistency in the orders of approximation in the first and second equations.

3. As was pointed out in [2], in a macroscopic description of a medium with microstructure, the acoustic region of vibrations, in which the displacement of the mass center is the fundamental kinematic variable, is of greatest interest. The special role of this variable is related to the circumstance that in the admissible acoustic region the system of equations (1.27) can be equivalently transformed into one equation containing only the single variable u and other equations which explicitly express the remaining kinematic

variables η in terms of u. In matrix notation Equations (1.27) coincide with the corresponding equations for the one-dimensional case, and, therefore, the general scheme of transformation of the equations is retained for the three-dimensional case. This permits giving the final results without dwelling on the details.

We introduce the matrix

$$A(k, \omega) = \|\Gamma(k) - \omega^2 I\|^{-1} = A^+(k, \omega) \tag{3.1}$$

As was done in [2] it is possible to show that $A(k, \omega)$ exists in some finite part of the acoustic region containing the origin of coordinates (the admissible acoustic region), and can be explicitly expressed in terms of the coefficients of the expansion of $\Gamma(k)$ in a series in k. With the aid of the matrix $A(k, \omega)$, the system (1.27) is transformed into

$$\omega^2 \rho g^{\alpha\beta} u_\beta - \psi^{\alpha\beta\lambda\mu}(k, \omega) k_\lambda k_\mu u_\beta = -Q^\alpha \tag{3.2}$$

$$\eta_{p\alpha} = i a^{\cdot\cdot\mu\beta}_{p\alpha}(k, \omega) k_\mu u_\beta + A_{p\alpha q\beta}(k, \omega) \mu^{q\beta} \tag{3.3}$$

where

$$\psi^{\alpha\beta\lambda\mu}(k, \omega) = \gamma^{\lambda\alpha\mu\beta}(k) - [\chi^{+\lambda\alpha p\nu}(k) A_{p\nu q\tau}(k, \omega) \chi^{q\tau\mu\beta}(k)]_{(\lambda\mu)}$$
$$a^{\cdot\cdot\mu\beta}_{p\alpha}(k, \omega) = A_{p\alpha q\tau}(k, \omega) \chi^{q\tau\mu\beta}(k) \tag{3.4}$$
$$Q^\alpha = q^\alpha + i k_\lambda \mu^{\lambda\alpha}, \qquad \mu^{\lambda\alpha} = -a^{+\lambda\alpha}_{\cdot\cdot q\beta}(k, \omega) \mu^{q\beta}$$

Equation (3.2) coincides in form with the equation of motion of an elastic medium with simple structure [1]. However, in the present case $\psi^{\alpha\beta\lambda\mu}$ depends not only on k but also on ω, i.e. in addition to the spatial distribution there exists a timewise dispersion which is obviously unrelated to any dissipation of energy. It is easy to show that for small k and ω the dispersion appears beginning with the second approximation.

The right-hand side of (3.2) contains the equivalent density of external forces Q^α, which is equal to the mean density of the external forces q^α minus the divergence of the density of the micro-moments $\mu^{\lambda\alpha}$, as in the ordinary macroscopic theory.

By an argument similar to that used in [1], it is possible to obtain an expression for the strain energy density and to define a symmetric stress tensor. To this end, we introduce the tensor (S is the symmetrization operator defined in [1])

$$c^{\lambda\alpha\mu\beta}(k, \omega) = S\psi^{\alpha\beta\lambda\mu}(k, \omega) \tag{3.5}$$

which is symmetric in the first pair of scripts, and we denote the elastic distortion by

$$\zeta^{\circ}_{\lambda\alpha}(x) = \partial_\lambda u_\alpha(x) = \varepsilon_{\lambda\alpha}(x) + \Omega_{\lambda\alpha}(x)$$

The strain energy density can be expressed in the form

$$\varphi(x) = {}^1/_2 \sigma^{\lambda\alpha}(x) \varepsilon_{\lambda\alpha}(x) \tag{3.6}$$

where the symmetric tensor $\sigma^{\lambda\alpha}$ can be interpreted as the stress tensor (*).

*) At the same time it does not appear advisable to introduce directly into Equations (1.28) a choice (generally an arbitrary one) of couple-stress tensors, since they are not well defined pgysical quantities. This follows from the analysis of the original microscopic model.

In the (k, ω) representation

$$\sigma^{\lambda\alpha}(k) = c^{\lambda\alpha\mu\beta}(k, \omega)\,\zeta_{\mu\beta}(k) \tag{3.7}$$

In order that (3.6) be invariant under rotation, it is necessary and sufficient to require that

$$\psi_0^{\alpha\beta\lambda\mu} = \psi_0^{\lambda\mu\alpha\beta} \tag{3.8}$$

Straightforward computation shows that (3.8) coincides with the condition (1.24). The relation (3.7) is the generalized Hooke's law. In the (x, t) representation, Hooke's law is written in the obvious operator form and the equations of motion assume the usual form

$$\operatorname{div}\sigma = \rho u^{\cdot\cdot} - Q \tag{3.9}$$

In the case of weak distribution and in the zeroth approximation with respect to k and ω, Equations (3.2) are transformed into the equations of the classical theory of elasticity, and $c_0^{\lambda\alpha\mu\beta} = c^{\lambda\alpha\mu\beta}(0, 0)$ coincides with the ordinary tensor of the elastic constants which are measured in macroexperiments. The relation between the $c_0^{\lambda\alpha\mu\beta}$ and the microscopic force constants can be found from (3.4) and (3.5)

$$c_0^{\lambda\alpha\mu\beta} = S\psi_0^{\alpha\beta\lambda\mu} \tag{3.10}$$

It is easy to show that (3.10) coincides with the corresponding formula given in [3]. It is interesting to note that (3.10) is considerably simplified if the condition (2.1) is satisfied, since in that case it is necessary to carry out the inversion of the matrix $\Gamma_0^{p\alpha q\beta}$. Equation (3.10) then takes the form

$$c_0^{\lambda\alpha\mu\beta} = S\,[\alpha\beta, \lambda\mu] \tag{3.11}$$

In proceeding to the succeeding approximations in Equation (3.2), it is advisable, as before, to distinguish between the cases of gyrotropic and non-gyrotropic media in Equation (3.2). Here it should be noted that the property of gyrotropy in the sense indicated is not invariant under the transformation of the equations which has been carried out above. Thus, for instance, if the condition (2.1) is satisfied and the interactions are paired, then the medium described by Equations (1.27) is, generally speaking, gyrotropic. However, it can be shown that upon transforming to Equation (3.2), the first approximation coincides with the zeroth, and only in Equations (3.3) does non-gyrotropy make its appearance.

In the general case of a gyrotropic medium, the operator $\mathbf{c}$ in the first approximation has the form

$$\mathbf{c}^{\lambda\alpha\mu\beta} = c_0^{\lambda\alpha\mu\beta} + c_1^{\lambda\alpha\mu\beta\tau}\partial_\tau \tag{3.12}$$

and correspondingly, for a non-gyrotropic medium in the second approximation $(\partial_t = \partial/\partial t)$

$$\mathbf{c}^{\lambda\alpha\mu\beta} = c_0^{\lambda\alpha\mu\beta} + c_2^{\lambda\alpha\mu\beta\tau\varkappa}\partial_\tau\partial_\varkappa + c_2'^{\lambda\alpha\mu\beta}\partial_t^2 \tag{3.13}$$

For an isotropic medium in the general case of strong distribution

$$c^{\varkappa\alpha\tau\beta}(k, \omega) = \lambda(k, \omega)\,\delta^{\varkappa\alpha}\delta^{\tau\beta} + \mu(k, \omega)\,(\delta^{\alpha\beta}\delta^{\varkappa\tau} + \delta^{\varkappa\beta}\delta^{\tau\alpha}) \tag{3.14}$$

where λ and μ are even analytic functions of k and ω. The evenness with respect to ω is obvious, and evenness with respect to k follows from the fact that there exist no isotropic tensors of odd order.

Equations (3.2) may be written in the following form in the (x, t) representation for the second approximation:

$$J\ddot{u} - (\mu_0 + \mu_2\Delta)\,\Delta u - [\lambda_0 + \mu_0 + (\lambda_2 + \mu_2)\Delta]\,\mathrm{grad}\ \mathrm{div}\ u = Q \qquad (3.15)$$

Here the quantity

$$J = \rho - \mu_2'\Delta - (\lambda_2' + \mu_2')\,\mathrm{graddiv} \qquad (3.16)$$

as the operator giving the effective inertia of the medium. Equation (3.15) agrees with the corresponding equation of [7]. In the static case, (3.15) coincides with the equation of equilibrium for a medium of simple structure and, therefore, the static Green's tensor, as given, e.g. in [1], may be used for its solution.

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